Actions of $\mathfrak{sl}_2$ on algebras appearing in categorifications

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1 $\mathfrak{sl}_2-$Categories

Theorem 1 (Main Theorem of EQ'22)
The following categories

- $\bigoplus_{\alpha \in Q^+_{A_{n-1}}} R(\alpha) - \text{mod}$ *(Cat of $U_q^+(\mathfrak{gl}_n)$)*
- $\dot{\mathcal{U}}(\mathfrak{sl}_2)$ *(Cat of $U_q(\mathfrak{sl}_2)$)*
- $\mathscr{D}(\mathbb{C}^n, S_n)$ *($\cong \mathbb{S}Bim(\mathbb{C}^n, S_n)$ in char 0) (Cat of $H_q(S_n)$)*

have the structure of a monoidal $\mathfrak{sl}_2-$cat.

Definition 1.1. *Let $\Bbbk$ be a commutative domain and $\mathfrak{g}$ a lie algebra over $\Bbbk$. A $\mathfrak{g}-$algebra is a $\Bbbk-$algebra A with an action of $\mathfrak{g}$ by derivations. We will write $(A, \mathfrak{g})$ for this structure.*

Example 1. Let $A = R_n = \Bbbk[x_1, \dots, x_n]$ where $\deg(x_i) = 2$ and let $\mathfrak{g} = \mathfrak{sl}_2 = \{d = e, h, -z = f\}$. Then

$$d = \sum_i x_i^2 \frac{\partial}{\partial x_i}, \qquad h \cdot p(x) = \deg(p(x))p(x), \qquad z = \sum_i \frac{\partial}{\partial x_i}$$

gives A the structure of a $\mathfrak{g}-$algebra (note that the weight grading agrees with the usual grading by construction). Note this is equivalent to $d(x_i) = x_i^2$ and $z(x_i) = 1 \ \forall i$ and extending by Leibniz rule.

Definition 1.2. *Let $\Bbbk$ be a commutative domain and $\mathfrak{g}$ a lie algebra over $\Bbbk$. A $\mathfrak{g}-$category $\mathscr{C}$ is a category with an action of $\mathfrak{g}$ on the morphism spaces such that composition of morphisms*

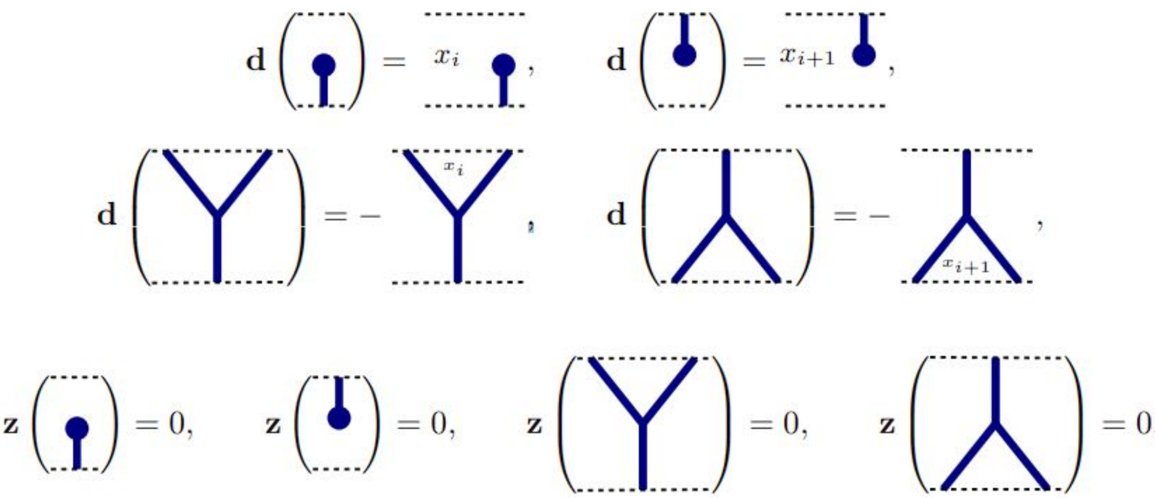
$$\text{Hom}_{\mathscr{C}}(A, B) \otimes \text{Hom}_{\mathscr{C}}(B, C) \to \text{Hom}_{\mathscr{C}}(A, C)$$

is a morphism of $\mathfrak{sl}_2-$modules. A monoidal $\mathfrak{g}-$category is a $\mathfrak{g}-$category which is also monoidal s.t.

$$x(f \otimes g) = x(f) \otimes g + f \otimes x(g)$$

(Note that composition of morphisms being a morphism of $\mathfrak{sl}_2-$modules is the same as above but with $\otimes$ replaced by $\circ$.)

Example 2. Let $\mathscr{C} = \mathscr{D}(\mathbb{C}^2, S_2)$ and $\mathfrak{g} = \mathfrak{sl}_2 = \{d = e, h, -z = f\}$. Hom spaces of $\mathscr{C}$ are monoidally generated by [draw trivalent and dots] as free R_2-modules. Define



and $h(\varphi) = \deg(\varphi)\varphi$. (Thus, the weight grading on the hom spaces agrees with the usual grading) For R_2, $\mathfrak{sl}_2$ acts on R_2 as above. To check that this indeed gives an action of $\mathfrak{sl}_2$, we need to check two things.

(1) $d, h, -z$ preserves the generating relations of $\mathscr{D}(\mathbb{C}^2, S_2)$

(2) $d, h, -z$ satisfies the relations of $\mathfrak{sl}_2$ when applied to the generating morphisms.

Let us check the barbell relation. [Do computation]. To check that $d, h, -z$ satisfies the relations of $\mathfrak{sl}_2$, d, z move us into the correct weight space by construction so we only need to check $[d, -z] = h$. z kills any diagram without polynomials and so we only need to check

$$z(d(\phi)) = h(\phi) = \deg(\phi)\phi$$

where ϕ is a generating morphism which we leave as an exercise.

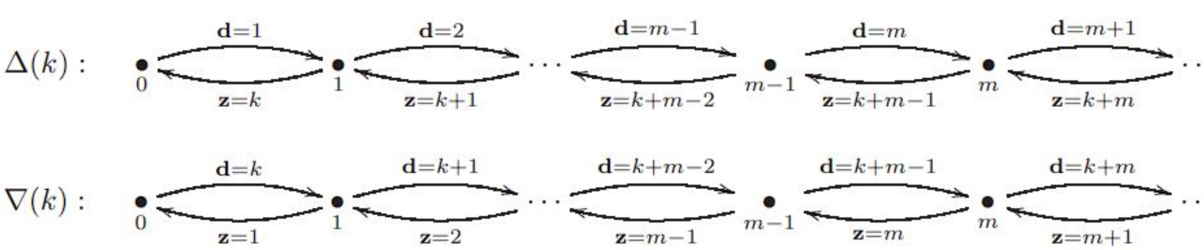
Remark. The notion of a $\mathfrak{g}$−algebra and a $\mathfrak{g}$−category can be viewed as analogues of dg−algebras and dg−categories where the former uses the hopf algebra $U(\mathfrak{sl}_2)$ while the latter uses the hopf algebra $\Bbbk[d]/(d^2)$.

(We want to now decompose hom spaces in the Soergel category as $\mathfrak{sl}_2$ representations, so we will now introduce the representations that show up.)

2 Representations of polynomial $\mathfrak{sl}_2$−algebras

2.1 Lowest Weight Vermas and CoVermas

Definition 2.1. *Again let $\Bbbk$ be a commutative domain. Given $k \in \Bbbk$, define the $\mathfrak{sl}_2$ modules (write $v_{k,0}$ instead of* 0 *for verma and $w_{k,0}$ for the coverma)*



(draw in action of $h(v_{k,m}) = (k+2m)v_{k,m}$) These are the lowest weight verma and coverma modules.

Proposition 2.2 (Properties). *(i) $\Delta(k), \nabla(k)$ are indecomposable for all $k \in \mathbb{Z}$.*

(ii) If $k > 0$, $\Delta(k), \nabla(k)$ are simple.

(iii) If $k \leq 0$ we have SES

$$0 \to \nabla(-k+2) \to \Delta(k) \to W(k) \to 0$$

$$0 \to W^{\vee}(k) \to \nabla(k) \to \Delta(-k+2) \to 0$$

where $W(k), W^{\vee}(k)$ are the Weyl and dual Weyl modules (Both equal L_k over $\mathbb{Q}$).

(iv) If $\Bbbk$ contains $\mathbb{Q}$ then $\Delta(k) \cong \nabla(k) \iff k > 0$. Otherwise $\nabla(k) \cong \Delta(k) \iff k = 1$.

Proof. (iii) Note that $-k+2 = s \bullet_{-\rho} k$, draw out when $m = -k+2$ for $\Delta(k)$. One might have expected that $\nabla(-k+2)$ and $\Delta(-k+2)$ swap places (this is what happens in category $\mathcal{O}$ for instance) but integrally by part (iv), $\nabla(-k+2) \neq \Delta(-k+2)$ most of the time. ■

2.2 Rank-one modules over polynomial rings

Recall that $R_1 = \Bbbk[x]$ and $d = x^2 \dfrac{\partial}{\partial x}$ and $z = \dfrac{\partial}{\partial z}$

Proposition 2.3. $R_1 \cong \nabla(0)$ as $\mathfrak{sl}_2 - \text{mod}$

Proof. $w_{0,m} \mapsto x^m$ is a clear bijection. Note that the action of d, z on $\nabla(0)$ is given by

$$d(w_{0,m}) = m w_{k,m+1} \qquad z(w_{0,m+1}) = (m+1) w_{0,m}$$

which exactly matches with $d(x^m) = m x^{m+1}, z(x^{m+1}) = (m+1)x^m$. ■

Definition 2.4. *Let $a \in \Bbbk$ and let $R_1 \langle a \rangle$ =free graded rank 1 R_1 module with generator 1_a. Define a $(R_1, \mathfrak{sl}_2)$ module structure on $R_1 \langle a \rangle$ via*

$$d(1_a) = ax \cdot 1_a, \qquad z(1_a) = 0, \qquad h(1_a) = a 1_a$$

and extending to all of $R_1 \langle a \rangle$ by the Leibniz rule.

Proposition 2.5. *(i) Any $(R_1, \mathfrak{sl}_2)-$module structure on R_1 where h acts semisimply is isomorphic to $R_1 \langle a \rangle$ for a unique $a \in \Bbbk$.*

(ii) We have an isomorphism $R_1 \langle a \rangle \cong \nabla(a)$ of $\mathfrak{sl}_2$ modules.

Proof. (i) Let M be free of rank 1 over R_1 with generator v. v is in lowest degree(weight). We know that $hv = cv$ for some $c \in \Bbbk$ by the semisimple condition. Now because the only way to increase the weight by 2 is to multiply by x, we must have that $d(v) = c'x \cdot v$ for some $c' \in \Bbbk$ and similarly $z(v) = 0$. Now applying $[d, -z] = h$ to v we have

$$cv = h(v) = (-dz + zd)(v) = z(c'x \cdot v) = c'v$$

and thus $c = c'$ and so the action of d, z, h matches with $R_1 \langle a \rangle$ above.
(ii) Left as an exercise to show that $x^m \cdot 1_a \mapsto w_{a,m}$ gives iso. ■

Definition 2.6. *Let $p(\vec{x}) = \sum a_i x_i$ be a linear polynomial in R_n and let $R_n \langle p(\vec{x}) \rangle$ =free graded rank 1 R_n-module with generator $1_{p(\vec{x})}$. Define a $(R_n, \mathfrak{sl}_2)$ module structure on $R_n \langle a \rangle$ via*

$$d(1_{p(\vec{x})}) = p(\vec{x}) \cdot 1_{p(\vec{x})}, \qquad z(1_{p(\vec{x})}) = 0, \qquad h(1_{p(\vec{x})}) = (\sum a_i) 1_{p(\vec{x})}$$

and extending via the Leibniz rule.

Proposition 2.7. *(i) Any $(R_n, \mathfrak{sl}_2)-$module structure on R_n where h acts semisimply is isomorphic to $R_n \langle p(\vec{x}) \rangle$ for some linear polynomial $p(\vec{x})$.*

(ii) We have an isomorphism of $\mathfrak{sl}_2$ modules,

$$R_n \langle p(\vec{x}) \rangle \cong R_1 \langle a_1 \rangle \otimes \ldots \otimes R_1 \langle a_n \rangle \cong \nabla(a_1) \otimes \ldots \otimes \nabla(a_n)$$

Example 3. Check that $\text{Hom}_{\mathbb{S}Bim(\mathbb{C}^2, S_2)}(R_2, B_s) = \nabla(0) \otimes \nabla(1)$. Also note that $\text{Hom}_{\mathbb{S}Bim(\mathbb{C}^1, S_2)}(R_1, R_1) = R_1 \cong \nabla(0)$ while $\text{Hom}_{\mathbb{S}Bim(\mathbb{C}^2, S_2)}(R_2, R_2) = R_2 \cong \nabla(0) \otimes \nabla(0)$. As you can see, the $\mathfrak{sl}_2$ module structure depends on the rank of the realization used.

Remark. In general, we know that $\text{Hom}_{\mathbb{S}Bim(\mathbb{C}^n, S_n)}(A, B)$ and in fact $\text{Ext}^\bullet_{\mathbb{S}Bim(\mathbb{C}^n, S_n)}(A, B)$ are free as left/right R_n-modules so one might naively hope that as $\mathfrak{sl}_2-$modules

$$\text{Hom}_{\mathbb{S}Bim(\mathbb{C}^n, S_n)}(A, B) \stackrel{?}{\cong} \bigoplus \bigotimes \nabla(a_{i_1}) \otimes \ldots \nabla(a_{i_n})$$

If this were true, one can imagine that by using the $\mathfrak{sl}_2$ action we can prove a lot of structural properties in the Soergel category but unfortunately this won't be true.

3 Filtrations

- (A $\mathfrak{sl}_2-$category is where $\mathfrak{sl}_2$ acts on morphism spaces)
- Main Theorem of EQ'22 = categories appearing in type A categorical representation theory are $\mathfrak{sl}_2-$categories.
- Any $(R_n, \mathfrak{sl}_2)-$module structure on R_n is isomorphic to $R_n \langle p(\vec{x}) \rangle \cong \nabla(a_1) \otimes \ldots \otimes \nabla(a_n)$ where $p(x) = \sum a_i x_i$ and $d(1_{p(\vec{x})}) = p(\vec{x}) \cdot 1_{p(\vec{x})}$.
- $\text{Hom}_{\mathbb{S}Bim(\mathbb{C}^n, S_n)}(A, B)$ is free as left/right R_n-module.

(Similarly, last semester Alvaro gave a sketch that $_{\vec{j}}R(\nu)_{\vec{i}}$ is free as an abelian group following [KL09]. The basis constructed had all dots at the bottom and so the proof also shows that $_{\vec{j}}R(\nu)_{\vec{i}}$ is free as an R_k-module, where $k = |\nu|$ is the number of strands. Again, the naive hope that morphism spaces is a direct sum of tensor product of covermas is wrong. HOWEVER, it turns out)

Definition 3.1. *Let M be a $(R_n, \mathfrak{sl}_2)-$module which is free and f.g. as a graded R_n-module so that $M = \oplus_{i \in I} M_i$ where each M_i is free of rank 1 over R_n. $\oplus_{i \in I} M_i$ is called a downfree filtration on M if*

(1) z preserves each M_i.

(2) $\exists$ partial order $\leq$ on I s.t. $d(M_i) \subset \oplus_{j \leq i} M_j$ for all $i \in I$

A homogeneous basis of M as an R_n-module is called downfree if it induces a downfree filtration on M.

Remark. Condition (2) above is similar to Sullivan algebras appearing in rational homotopy theory.

Given any poset $(I, \leq)$, we can always refine it to a (nonunique) total ordering $(I, \overset{\star}{\leq})$ such that $i \leq j \implies i \overset{\star}{\leq} j$.

Example 4. The Bruhat order on S_3 can be refined to $e < s < t < st < ts < sts$.

It follows that $\bigoplus_{j \overset{\star}{\leq} i} M_j$ are $(R_n, \mathfrak{sl}_2)$ submodules of M that gives a filtration on M with quotients that are free of rank 1 over R_n.

Definition 3.2. *Let M be an $(R_n, \mathfrak{sl}_2)-$module with a downfree filtration. Each subquotient must be isomorphic to $R_n \langle p_i(\vec{x}) \rangle$ for a unique $p_i(\vec{x}) \in R_n$. The multiset of linear polynomials $\{p_i(\vec{x})\}_{i \in I}$ is called the downfree character of M with respect to the filtration.*

Remark. The downfree filtration and character are analogues of the $\Delta-$filtration and characters for Soergel Bimodules.

Theorem 2 (Real Main Theorem of EQ'22)
The morphism spaces with corresponding poset in the following categories

- $\bigoplus_{\alpha \in Q^+_{A_{n-1}}} R(\alpha) - \text{mod}$ *(Bruhat Order)*
- $\mathscr{D}(\mathbb{C}^n, S_n)$ *(LexicoBruhat order on CoTerminal Bruhat Strolls)*

are downfree with basis given by (crossings)permutations in $S_{|\alpha|}$ and the double leaves basis, respectively. An explicit computation of their downfree characters is also given.[a]

[a] For double leaves, it's a decorated version of Deodhar's formula.

Example 5. We have that $\text{Hom}_{\mathscr{D}(\mathbb{C}^2, S_2)}(B_s, B_s) = R_2 \cdot$ $\oplus R_2 \cdot$  with poset $<$ . Compute $d\Big($  $\Big)$ and $d\Big($  $\Big)$. We therefore have the filtration of $(R_2, \mathfrak{sl}_2)-$modules

$$0 \subset R_2 \cdot$$ $$\subset \text{Hom}_{\mathscr{D}(\mathbb{C}^2, S_2)}(B_s, B_s)$$

is downfree with downfree character $\{0, x_1 + x_2\}$ and

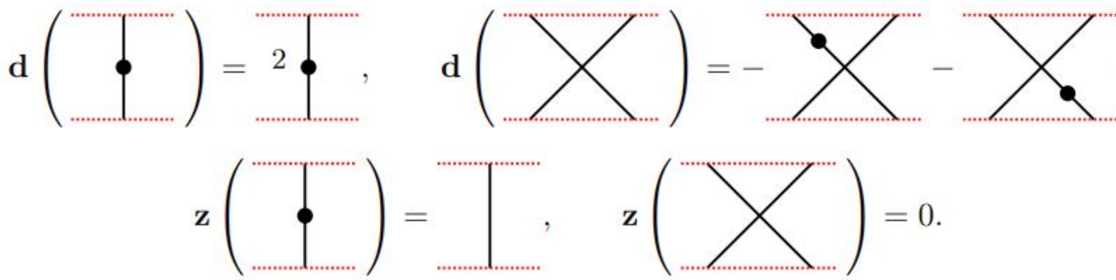
$$R_2 \cdot$$ $$\cong \nabla(0) \otimes \nabla(0), \qquad \text{Hom}_{\mathscr{D}(\mathbb{C}^2, S_2)}(B_s, B_s)/R_2 \cdot$$ $$\cong \nabla(1) \otimes \nabla(1)$$

This filtration actually splits since $R_2 \cdot$  is also a $\mathfrak{sl}_2$ submodule.

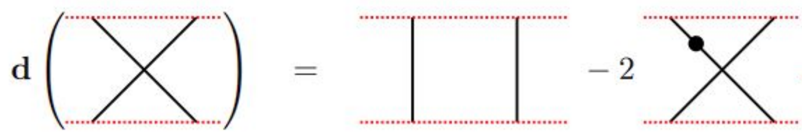
Example 6. Let $n = 2$, then $KLR(A_1) = \bigoplus_{m \geq 0} R(m\alpha) \cong \bigoplus_{m \geq 0} NH_m$. We can think of this as a $\Bbbk-$linear monoidal category $\mathscr{N}$ by setting $\bullet$ as the generating object with generating morphisms a single crossing and dots and define $NH_m = \text{End}_{\mathscr{N}}(m, m)$.The Nilhecke relation

[Draw]

allows us to slide all dots to the top or to the bottom. As a result there is a left(=top) and right (=bottom) action of R_m where x_1 corresponds to the first strand on top, etc. It is a Theorem of [KL09] that NH_m is free as a left or right R_m-module with basis indexed by permutations in S_m. For example, $NH_2 = R_2 \cdot \mathrm{id} \oplus R_2 \cdot X$ with poset $\mathrm{id} < X$. Define



Now using the NilHecke relation, we have that



We therefore have the filtration of $(R_2, \mathfrak{sl}_2)-$modules

$$0 \subset R_2 \cdot \mathrm{id} \subset NH_2$$

is downfree with downfree character $\{0, -2x_1\}$ and

$$R_2 \cdot \mathrm{id} \cong \nabla(0) \otimes \nabla(0), \qquad NH_2/R_2 \cdot \mathrm{id} \cong \nabla(-2) \otimes \nabla(0)$$

and now the filtration doesn't split.

4 Core

Definition 4.1. *Let* $\Bbbk$ *be char 0 and Noetherian. Let* M *be a bounded(weights bounded below, wt spaces finite rank) weight module for* $\mathfrak{sl}_2$*. Let*

$$\mathrm{Core}(M) = \left\{m \in M \mid d^N(m) = 0 \text{ for } N >> 0\right\}$$

Then $\mathrm{Core}(M)$ *will be the maximal submodule of* M *that is f.g. over* $\Bbbk$*.*

Example 7. $\mathrm{Core}(\nabla(a_i)) = W^\vee(a_i)$ if $a_i \leq 0$, otherwise $\mathrm{Core}(\nabla(a_i)) = 0$.

Proposition 4.2. *Let* $p(x) = \sum a_i x_i \in R_n$ *and* $a_i \in \mathbb{Z}$*. Then*

(i) If $a_i > 0$ *for some* i*,* $\mathrm{Core}(R_n \langle p(x) \rangle) = 0$*.*

(ii) If $a_i \leq 0$ $\forall i$ *then* $\mathrm{Core}(R_n \langle p(x) \rangle) \cong W^\vee(a_1) \otimes \cdots \otimes W^\vee(a_n)$

Proof. *(ii)* Clearly $W^\vee(a_1) \otimes \cdots \otimes W^\vee(a_n) \subseteq \mathrm{Core}(R_n \langle p(x) \rangle)$ by 2nd definition of the core. As $a_i \leq 0$ $\forall i$, $\nabla(a_i)/W^\vee(a_i) = \Delta(-a_i + 2)$. Therefore $R_n \langle p(x) \rangle / W^\vee(a_1) \otimes \cdots \otimes W^\vee(a_n)$ has a filtration with subquotients isomorphic to $M \otimes \Delta(a)$ for some a. Note $d^N(v) \neq 0 \;\; \forall v \in \Delta(a)$. If d has no nilpotents on the associated graded, then it has no nilpotents on the entire module, and so by the first definition of the core, $\mathrm{Core}(R_n \langle p(x) \rangle / W^\vee(a_1) \otimes \cdots \otimes W^\vee(a_n)) = 0$ giving the reverse inclusion. ■

4.1 Decompositions

Let $\mathrm{gJac}(A)$ be the graded Jacobson radical of an algebra A.

Example 8. $\mathrm{gJac}(\Bbbk[x]) = (x)$. HOWEVER, notice that under the $\mathfrak{sl}_2-$isomorphism $\Bbbk[x] \cong \nabla(0)$, we have that $\Bbbk \cdot 1 \cong W^{\vee}(0)$, aka the compliment of $\mathrm{gJac}(\Bbbk[x])$ equals $\mathrm{Core}(\nabla(0))$. Somehow, the $\mathfrak{sl}_2$ structure "knows" about the algebra structure.

Definition 4.3. *Let $\mathscr{C}$ be an additive (graded) category. Given $X, Y \in \mathscr{C}$, define*

$$\begin{aligned} \mathrm{rad}(X,Y) &= \{f \in \mathrm{Hom}_{\mathscr{C}}(X,Y) \,|\, \forall g \in \mathrm{Hom}_{\mathscr{C}}(Y,X)\ \mathrm{id}_X - gf \text{ is invertible in } \mathrm{End}_{\mathscr{C}}(X)\} \\ &= \{f \in \mathrm{Hom}_{\mathscr{C}}(X,Y) \,|\, \forall g \in \mathrm{Hom}_{\mathscr{C}}(Y,X)\ \mathrm{id}_Y - fg \text{ is invertible in } \mathrm{End}_{\mathscr{C}}(Y)\} \end{aligned}$$

Now define the (graded) Jacobson radical of $\mathcal{C}$ to be

$$\mathscr{J}ac(\mathcal{C}) := \bigoplus_{X,Y \in \mathcal{C}} \mathrm{rad}(X,Y)$$

Remark. $\mathrm{rad}(X,X) = \mathrm{gJac}(\mathrm{End}_{\mathcal{C}}(X,X))$.

Conjecture 3
For the categories $\mathscr{C}$ above, $\mathrm{Core}(\mathscr{C}) = \bigoplus_{X,Y \in \mathscr{C}} \mathrm{Core}(\mathrm{Hom}_{\mathscr{C}}(X,Y))$ intersects $\mathscr{J}ac(\mathcal{C})$ trivially.

Corollary 4.4. *Assuming Conjecture 3, if B is indecomposable, then any injection $i \in \mathrm{Core}(\mathrm{Hom}_{\mathscr{C}}(B,X))$ splits. Similarily with surjections in $\mathrm{Core}(\mathrm{Hom}_{\mathscr{C}}(X,B))$.*

Proof. $\mathrm{Core}(\mathrm{Hom}_{\mathscr{C}}(B,X)) \cap \mathrm{rad}(B,X) = 0$ means that $\exists g$ s.t. $1 - g \circ i$ is not invertible. B indecomposable means $\mathrm{End}(B)$ is a local ring and so $1 - (1 - g \circ i) = g \circ i$ is invertible so i splits. ∎

Lemma 4.5. *Let M be a bounded $\mathfrak{sl}_2-$representation. Then*

$$\mathrm{Core}(M) = \ker d + z \cdot \ker d + z^2 \cdot \ker d + \dots$$

This process terminates in a finite number of steps because M is bounded.

Corollary 4.6. *Assuming Conjecture 3, if $i \in \mathrm{Hom}_{\mathscr{C}}(B,X)$ is an inclusion(projection) map and d acts nilpotently on i, then $z^k(i), d^k(i)$ for $k \geq 2$ are also inclusion(projection) maps.*

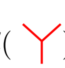
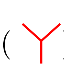
Example 9. In $\mathbb{S}Bim(\mathbb{C}^2, S_2)$

$$B_s \otimes_{R_2} B_s \cong B_s(-1) \oplus B_s(1)$$

(This underlies Reidemeister II invariance in triply graded link homology) $\mathrm{Hom}(B_s, B_s B_s^2)$ has a left R_2 basis given by the double leaves

$$\begin{pmatrix} \epsilon\epsilon \\ \eta \end{pmatrix}, \begin{pmatrix} \epsilon \circ \mu \\ \eta \end{pmatrix}, \begin{pmatrix} \epsilon & id \\ & id \end{pmatrix}, \begin{pmatrix} \mu \\ id \end{pmatrix}$$

Let us try to find the core. Applying d to the first 3 maps will create polynomials forever. However, note that

Compute $d($  $)$ and $d^2($  $)$

and in fact these are the inclusion maps for $B_s(1)$ and $B_s(-1)$, respectively. For the experts, note that $\Delta(1) = x_1 \otimes_s 1 - 1 \otimes_s x_2$ in the $\mathfrak{gl}_2$ realization of S_2.

Warning. The core doesn't always contain all the inclusion/projection maps!

Example 10. In $\mathbb{S}Bim(\mathbb{C}^3, S_3)$

$$B_s \otimes_{R_3} B_t \otimes_{R_3} B_s \cong B_{sts} \oplus B_s$$

The projection map p_s for B_s will be pitchfork. The projection map p_{sts} for B_{sts} will be the Jones-Wenzl projector. We have that $d(p_s) = 0$, but $d(p_{sts}) \neq 0$ (One needs to define d on thick soergel calculus first). In fact $\text{Core}(\text{Hom}(B_sB_tB_s, B_{sts})) = 0$! However,

- $d(p_{sts}) \in \text{Hom}(B_s, B_{sts}) \circ p_s$
- $\text{Core}(\text{Hom}(B_sB_tB_s, B_{sts})/(\text{Hom}(B_s, B_{sts}) \circ p_s)\,) = \Bbbk \cdot p_{sts}$

In fact, the opposite is true, namely

- $d(i_{sts})(= d(JW) \circ JW?) = 0$
- $d(i_s) \in i_{sts} \circ \text{Hom}(B_s, B_{sts})$
- $\text{Core}(\text{Hom}(B_s, B_sB_tB_s)/(i_{sts} \circ \text{Hom}(B_s, B_{sts}))\,) = \Bbbk \cdot i_s$

In general one needs to take <u>iterated cores</u> in order to find enough projection/inclusion maps to split X into indecomposables.

Definition 4.7. *Let $\mathcal{C}$ be an $\mathfrak{sl}_2-$category. The $\mathfrak{sl}_2-$enriched category $\mathcal{C}_{enrich}$ will be an $\mathfrak{sl}_2-$category defined as follows.*

- ***Objects:*** *$B \boxtimes V$ where $B \in \mathcal{C}$ and $V \in \text{Rep}_{f.d.}\mathfrak{sl}_2$*
- ***Morphisms:***

$$\text{Hom}_{\mathcal{C}_{enrich}}(B \boxtimes V, B' \boxtimes V') := \text{Hom}_{\mathcal{C}}(B, B') \otimes \text{Hom}_{\mathfrak{sl}_2}(V, V')$$

Remark. There will be a monoidal action of Rep $\mathfrak{sl}_2$ on $\mathcal{C}_{enrich}$, where on objects one has

$$(B \boxtimes V) \otimes V' = B \boxtimes (V \otimes V')$$

As a consequence, it follows that the Grothendieck ring $K_\Delta(\mathcal{C}_{enrich})$ will be a module over $K_0(\text{Rep }\mathfrak{sl}_2) = \Bbbk[\mathbb{Z}]^{S_2} = \Bbbk[q + q^{-1}]$. Therefore, given a basis for $K_\Delta(\mathcal{C}_{enrich})$ over $K_0(\text{Rep }\mathfrak{sl}_2)$, structure constants will be unimodal polynomials.